Math Extra Credit 2

December 11, 2011

1 Microwaving water(100pts.)

You have a 1200W microwave. When you put a cup of water in only half evaporates in 8 minutes. How efficient is your microwave?

The heat needed to bring the water from room temperature to boiling and then boil half of it is

$$Q_{\text{tot}} = Mc\triangle T + mL =$$
$$.2366\text{kg} \cdot 4186\frac{\text{J}}{\text{kg}\cdot{}^0\text{C}} \cdot \left(100^0\text{C} - 25^0\text{C}\right) + \frac{.2366\text{kg}}{2} \cdot 2260000\frac{\text{J}}{\text{kg}} = 341639\text{J}$$

It delivers this heat energy to the water in 8 minutes making the power delivered simply

$$P = \frac{Q_{\text{tot}}}{8\text{min}} = \frac{341639\text{J}}{480\text{s}} = 711.748\text{W}$$

Therefore the efficiency of the microwave is

$$e = 100 \cdot \frac{711.748\text{W}}{1200\text{W}} = 59.3123\% \approx 60\%\surd$$

2 Ideal Gas(110pts.)

A car tire is pressurized in the morning at a temperature of 20^0C to 2atm. By afternoon at a temperature of 35^0C the tire pressure has changed to 2.1atm, but the volume has decreased by 2%. Use the ideal gas law to show that the reason this happened is because of a leak in the tire.

Using the ideal gas law which is $PV = nRT$ we can make use of the gas constant R which is not changing during this process and claim

$$\left(\frac{PV}{nT}\right)_{\text{morning}} = \left(\frac{PV}{nT}\right)_{\text{afternoon}}$$
$$\frac{2\text{atm}\cdot V}{n\cdot 293\text{K}} = \frac{2.1\text{atm}\cdot .98V}{n'\cdot 308\text{K}}$$
$$\frac{2\text{atm}}{n\cdot 293\text{K}} = \frac{2.1\text{atm}\cdot .98}{n'\cdot 35^0\text{C}}$$

Solve for the ratio of initial to final moles

$$\frac{n'}{n} = \frac{293\text{K}\cdot 2.1\text{atm}\cdot .98}{2\text{atm}\cdot 308\text{K}} = .978886 \approx .98\surd$$

There is only 98% of air left, it lost some air to a leak.

3 Molecular Speeds(100pts.)

The air around us is roughly 80% diatomic nitrogen and 20% diatomic oxygen. Which of these is faster and by how much?

Solve the equipartion kinetic energy relation for v.

$$\frac{1}{2}mv^2 = \frac{3}{2}kT$$

$$v = \sqrt{\frac{3kT}{m}}$$

The mass of an N_2 molecule is 28amu so its speed is

$$v_{\mathrm{N}} = \sqrt{\frac{3 \cdot 1.38 \cdot 10^{-23} \frac{\mathrm{J}}{\mathrm{K}} \cdot 300K}{28 \cdot 1.66 \cdot 10^{-27} \mathrm{kg}}} = 517.112 \frac{\mathrm{m}}{\mathrm{s}}$$

The mass of an O_2 molecule is 16amu so its speed is

$$v_{\mathrm{O}} = \sqrt{\frac{3 \cdot 1.38 \cdot 10^{-23} \frac{\mathrm{J}}{\mathrm{K}} \cdot 300K}{32 \cdot 1.66 \cdot 10^{-27} \mathrm{kg}}} = 483.714 \frac{\mathrm{m}}{\mathrm{s}}$$

The difference is $v_{\mathrm{d}} = 517.112 \frac{\mathrm{m}}{\mathrm{s}} - 483.714 \frac{\mathrm{m}}{\mathrm{s}} \approx 33 \frac{\mathrm{m}}{\mathrm{s}} \surd$

A Nitrogen molecule is faster by 33m/s.

4 Rydberg Formula(120pts.)

Hydrogenic atoms in an ultraviolet light fall from quantum energy state $n = 9$ to state $n = 2$ while releasing ultraviolet photons. The light has a power rating of 100W. How many photons are being released per second?

The frequency is given by

$$\frac{1}{\lambda} = \frac{f}{c} = R \left(\frac{1}{n_i^2} - \frac{1}{n_{\mathrm{f}}^2} \right)$$

$$f = 3 \cdot 10^8 \frac{\mathrm{m}}{\mathrm{s}} \cdot 1.097 \cdot 10^7 \frac{1}{\mathrm{m}} \left(\frac{1}{2^2} - \frac{1}{9^2} \right) = 7.8212 \cdot 10^{14} \frac{1}{\mathrm{s}}$$

The energy of the photon emitted is

$$E = hf = 6.626 \cdot 10^{-34} \mathrm{J} \cdot \mathrm{s} \cdot 7.8212 \cdot 10^{14} \frac{1}{\mathrm{s}} = 5.18233 \cdot 10^{-19} \mathrm{J}$$

The power of the light is

$$P = 100 \frac{\mathrm{J}}{\mathrm{s}} = n \cdot \frac{E}{t}$$

Solving for number released per time, $\frac{n}{t}$ we get

$$\frac{n}{t} = \frac{100\mathrm{J}}{5.18233 \cdot 10^{-19} \mathrm{J} \cdot \mathrm{s}} = 1.92963 \cdot 10^{20} \frac{\mathrm{photons}}{\mathrm{s}} \surd$$

5 Quantum Mechanics(300pts.)

Solve the time-independent Schrodinger equation for a particle of mass m in a box of width a. Then find the energies.

The equation is

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} = E\psi$$

Assume a solution similar to the free particle with boundary conditions at the edge of the box.

$$\psi\left(x\right) = A\cos kx + B\sin kx$$

Impose the boundary conditions because at the walls the particle has no chance of being there as the potential is infinite. The left side of the box gives us

$$\psi\left(0\right) = A\cos 0 + B\sin 0 = 0$$

From this we set $A = 0$ since $\cos 0 \neq 0$. The right side of the box gives us

$$\psi\left(a\right) = B\sin ka = 0$$

This tells us $\sin ka = 0$ which is only satisfied when $k = \frac{n\pi}{a}$ for $n = 0, 1, 2, ...$ Giving us a solution of

$$\psi\left(x\right) = B\sin\frac{n\pi}{a}x$$

Now to normalize wave function for correct probability we use

$$\int_0^a B\sin\frac{n\pi}{a}xB\sin\frac{m\pi}{a}xdx = 1$$

$$B^2\int_0^a \sin\frac{n\pi}{a}x\sin\frac{m\pi}{a}xdx = B^2\frac{a}{2} = 1$$

So $B = \sqrt{\frac{2}{a}}$ giving us the final normalized solution to the particle in a box of

$$\psi\left(x\right) = \sqrt{\frac{2}{a}}\sin\frac{n\pi}{a}x\surd$$

To find energies we simply plug it back in to the Schrodinger equation

$$-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\left(\sqrt{\frac{2}{a}}\sin\frac{n\pi}{a}x\right) = E\sqrt{\frac{2}{a}}\sin\frac{n\pi}{a}x$$

$$-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\sin\frac{n\pi}{a}x = E\sin\frac{n\pi}{a}x$$

$$\frac{\hbar^2}{2m}\frac{n^2\pi^2}{a^2}\sin\frac{n\pi}{a}x = E\sin\frac{n\pi}{a}x$$

$$E = \frac{\hbar^2}{2m}\frac{n^2\pi^2}{a^2}\surd$$

6 Periodic Table(90pts.)

Which two atoms would have a bond that is most ionic in character?

Lithium	Carbon
Cesium	Copper
Potassium	Nitrogen
Sodium	Chlorine

The best way to do this is with the Pauing electronegativity scale. The values of each element are

Lithium .98	Carbon 2.55
Cesium .79	Copper 1.9
Potassium .82	Nitrogen 3.04
Sodium .93	Chlorine 3.16

Now take the difference of the combined values

$$|.98\text{-}2.55| = 1.57$$

$$|.79\text{-}1.9| = 1.11$$

$$|.82\text{-}3.04| = 2.22$$

$$|.93\text{-}3.16| = 2.23$$

So it looks like Sodium Chloride forms the most ionic bond.

7 Solid State structures(200pts.)

Which has a greater packing fraction, a diamond crystal structure or a copper crystal structure? In other words, if every atom were to be a hard sphere just big enough to touch adjacent spheres, how much empty space would be left in each structure?

In diamond there are

- 8 spheres on the 8 corners of the unit cubic cell, only 1/8 of the volume is in the cell
- 6 spheres on the 6 faces of the unit cubic cell, only 1/2 of the volume is in the cell
- 4 spheres completely within the unit cell, the whole volume is in the cell

The unit cell is a cube with length 1. The spheres that are completely within the cell are at $\left(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}\right)$. As we left the point sphere expand to touch the adjacent neighbor spheres, these are the first hard sphere to touch with the corner spheres. This means our sphere radius is
$R = \frac{1}{2}\sqrt{\left(\frac{1}{4}\right)^2 + \left(\frac{1}{4}\right)^2 + \left(\frac{1}{4}\right)^2} = \frac{\sqrt{3}}{8}$

Now to count the amount of sphere within the cell we use the above to get

$$8 \cdot \frac{1}{8} + 6 \cdot \frac{1}{2} + 4 = 8$$

So the packing fraction is $8 \cdot \frac{4}{3}\pi \left(\frac{\sqrt{3}}{8}\right)^3 = .340087 \approx 34\% \surd$

In copper there are

- 8 spheres on the 8 corners of the unit cubic cell, only 1/8 of the volume is in the cell
- 6 spheres on the 6 faces of the unit cubic cell, only 1/2 of the volume is in the cell

The spheres can expand out to a radius of

$$R = \frac{1}{4}\sqrt{1^2 + 1^2} = \frac{\sqrt{2}}{4}$$

The total spheres that fill the unit cell volume is

$$8 \cdot \frac{1}{8} + 6 \cdot \frac{1}{2} = 4$$

So the packing fraction is $4 \cdot \frac{4}{3}\pi \left(\frac{\sqrt{2}}{4}\right)^3 = .74048 \approx 74\% \surd$

The close-packed copper structure with hard spheres has a much higher packing fraction..

8 Nuclear Energy(130pts.)

Because it has the highest binding energy per nucleon of all nuclides, ${}^{62}_{28}\mathrm{Ni}$ may be described as the most strongly bound. Its neutral atomic mass is 61.928349amu. Find its mass defect, total binding energy and its binding energy per nucleon.

Ths mass defect is the difference between the mass of the nucleus and the combined mass of its constituent nucleons. The binding energy E_{B} is this quantity multiplied by c^2, and the binding energy per nucleon is E_{B}divided by the mass number A .

The mass defect for nickel is

$$ZM_{\mathrm{H}} + Nm_{\mathrm{n}} - {}^{A}_{Z}M =$$
$$28 \cdot 1.007825\mathrm{amu} + (62 - 28)\,1.008665\mathrm{amu} - 61.928349 = .585361\mathrm{amu}$$

The binding energy is then just

$$.585361\mathrm{amu} \cdot c^2 = .585361\mathrm{amu} \cdot 931.5\frac{\mathrm{MeV}}{\mathrm{amu}} = 545.3\mathrm{MeV}$$

The binding energy per nucleon is then this divided by 62 so, 8.795 MeV per nucleon.

9 Radioactivity(110pts.)

Phil finds an old bone at an archeological site and decides to date it. The site is located geographically at a position where the ratio of ^{14}C to ^{12}C in the atmosphere before the year 1900 was about $1.31 \cdot 10^{-12}$. A small sample from the bone has a ratio if about $3.41 \cdot 10^{-13}$. The half-life of ^{14}C is 5730y. How old is this bone?

Using the decay formula $N = N_0 e^{-\lambda t}$ we solve for time t.

$$t = \frac{\ln\left(\frac{N}{N_0}\right)}{-\lambda}$$

Now we must find λ. It is

$$\lambda = \frac{\ln 2}{5730\text{y}} = 1.209 \cdot 10^{-4} \frac{1}{\text{y}}$$

Now plug in the appropriate values

$$t = \frac{\ln\left(\frac{3.41 \cdot 10^{-13}}{1.31 \cdot 10^{-12}}\right)}{-1.209 \cdot 10^{-4} \frac{1}{\text{y}}} = 11,132.3\text{y} \approx 11,100\text{y}\checkmark$$

10 Astrophysics(90pts.)

A number of galaxies have supermassive black holes at their centers. As material swirls around such a black hole, it is heated, becomes ionized, and generates a strong magnetic field. The resulting magnetic forces steer some of the material into highspeed jets that blast out of the galaxy and into intergalactic space. The blue light we observe from the jet has a frequency of $6.66 \cdot 10^{14}$Hz, but in the frame of reference of the jet material the light has a frequency of $5.55 \cdot 10^{13}$Hz. At what fraction of the speed of light is the jet moving toward us?

We must solve the relativistic Doppler formula for v.

$$f = f_0 \sqrt{\frac{c+v}{c-v}}$$

$$v = c \frac{\left(\frac{f}{f_0}\right)^2 - 1}{\left(\frac{f}{f_0}\right)^2 + 1} = .986c$$