Mathematical Extra Credit

December 11, 2011

1 Equations of motion(100pts.)

A cannon that is 10m long is designed to launch a 10kg ball over a castle wall. In order to do this the ball must have a speed of at least 50m/s as it exits the cannon. For every 10kg of explosives used, the force on the ball in the cannon increases by 1000N. How many kg of explosives should they use?

It must accelerate to 50m/s according to this equation

$$V_{\mathrm{f}}^2 = V_{\mathrm{i}}^2 + 2a\triangle x$$

$$V_{\mathrm{f}}^2 = 0 + 2a\triangle x$$

$$a = \frac{V_{\mathrm{f}}^2}{2\triangle x}$$

So the force on the ball has to be according to Newton's 2nd law

$$F = ma = m\frac{V_{\mathrm{f}}^2}{2\triangle x} = 10\mathrm{kg}\frac{\left(50\frac{\mathrm{m}}{\mathrm{s}}\right)^2}{2\cdot 10\mathrm{m}} = 1250\mathrm{kg}\frac{\mathrm{m}}{\mathrm{s}^2} = 1250\mathrm{N}.$$

Which means they need 12.5kg$\surd$ of explosives.

2 Gravitation(100pts.)

You throw a bowling ball off the empire state building(381m high) 10m/s downward. Neglecting air resistance, how long does it take to get to the bottom?

$$\triangle x = V_{\mathrm{i}}t + \frac{1}{2}gt^2$$

$$-381\mathrm{m} = -10\frac{\mathrm{m}}{\mathrm{s}}t + \frac{1}{2}\left(-9.8\frac{\mathrm{m}}{\mathrm{s}^2}\right)t^2$$

$$4.9\frac{\mathrm{m}}{\mathrm{s}^2}t^2 + 10\frac{\mathrm{m}}{\mathrm{s}}t - 381\mathrm{m} = 0$$

Solve this using the quadratic formula

$t = \frac{-b\pm\sqrt{b^2-4ac}}{2a} = \frac{-10\frac{\mathrm{m}}{\mathrm{s}}\pm\sqrt{\left(10\frac{\mathrm{m}}{\mathrm{s}}\right)^2-4\cdot 4.9\frac{\mathrm{m}}{\mathrm{s}^2}\cdot(-381\mathrm{m})}}{2\cdot 4.9\frac{\mathrm{m}}{\mathrm{s}^2}} \approx 7.9\mathrm{s}\surd$

Note: The quadratic formula has two solutions, $t_1 = -9.89714\mathrm{s}$ and $t_2 = 7.85632\mathrm{s}$.

3 Electromagnetic Fields(150pts.)

A singly ionized molecule with mass of 1000amu is released from rest on a plate with potential 0V. Another singly ionized molecule with mass of 4000amu on the same plate is released from rest at the same time. 1m away is a plate with potential of 10V. Which mass gets to the 10V plate first and by how much time does it win?

The electric field is calculated with $E = \frac{V}{m} = \frac{10\text{V}}{1\text{m}}$

Equate the force laws to calculate acceleration.

$$F = qE = ma$$

$$a = \frac{qE}{m}$$

The first mass accelerates at

$$a_1 = \frac{qE}{m_1} = \frac{1.6\cdot10^{-19}\text{C}\cdot10\frac{\text{V}}{\text{m}}}{1000\cdot1.66\cdot10^{-27}\text{kg}} = 963855\frac{\text{m}}{\text{s}^2}$$

while the second accelerate at

$$a_2 = \frac{qE}{m_2} = \frac{1.6\cdot10^{-19}\text{C}\cdot10\frac{\text{V}}{\text{m}}}{4000\cdot1.66\cdot10^{-27}\text{kg}} = 2.5\frac{\text{m}}{\text{s}^2}$$

The time it takes to get to the second plate is found using

$$\triangle x = V_i t + \frac{1}{2}at^2$$

$$2\text{m} = 0 + \frac{1}{2}at^2$$

$$t = \sqrt{\frac{2\text{m}}{a}}$$

The small mass gets to the second plate in

$$t_1 = \sqrt{\frac{2\text{m}}{a_1}} = .00144049\text{s} \approx 1.4\text{ms}$$

The bigger mass gets to the second plate in

$$t_2 = \sqrt{\frac{2\text{m}}{a_2}} = .00288097\text{s} \approx 2.9\text{ms}$$

So the small mass wins by 1.5ms$\surd$

4 Projectile motion(150pts.)

A baseball is crushed from 1m high at an angle of 36 degrees above the ground. It just barely made it over the center field wall which is 3m tall and 120m away. What was the initial speed?

The ball gets to the wall in an amount of time t defined by

$$V \cos\theta = \frac{d}{t}$$

$$\text{so } t = \frac{d}{V\cos\theta}$$

The vertical motion is now defined by

$$\triangle y = V \sin\theta t - \frac{1}{2}at^2$$

substitute t in

$$\triangle y = V \sin\theta \frac{d}{V\cos\theta} + \frac{1}{2}a\left(\frac{d}{V\cos\theta}\right)^2$$

Now solve for V

$$\triangle y = \tan\theta d + \frac{1}{2}a\frac{d^2}{V^2\cos^2\theta}$$

$$\frac{1}{2}a\frac{d^2}{V^2\cos^2\theta} = \triangle y - \tan\theta d$$

$$\frac{ad^2}{2(\triangle y - \tan\theta d)} = V^2\cos^2\theta$$

$$V = \sqrt{\frac{ad^2}{2(\triangle y - \tan\theta d)\cos^2\theta}} = \sqrt{\frac{-9.8\frac{\text{m}}{\text{s}^2}(120\text{m})^2}{2(2\text{m} - \tan 36^0 120\text{m})\cos^2 36^0}} = 35.5746\frac{\text{m}}{\text{s}} \approx 81\text{mph}\surd$$

5 Archimedes Principle(100pts.)

Franky claims he has a mass of 65kg. He is sitting on a spherical bouy that floats just barely submerged under water. When he gets off it rises halfway out of the water. What must the radius R in centimeters of the sphere be for him to be telling the truth?

According to Archimedes principle the bouyancy force necessary to keep Franky afloat is equal to his weight, or stated mathematically

$$F = mg = \rho V g$$

$$m = \rho V$$

The volume of the displaced fluid was half a sphere so

$$V = \frac{m}{\rho} = \frac{1}{2}\frac{4}{3}\pi R^3$$

Now solve for R

$$R = \sqrt[3]{\frac{3}{2\pi}\frac{m}{\rho}}$$

Now the density of the displaced fluid is water's density, $\rho = 1000\frac{\text{kg}}{\text{m}^3}$ and $m = 65\text{kg}$ so

$$R = \sqrt[3]{\frac{3}{2\pi}\frac{65\text{kg}}{1000\frac{\text{kg}}{\text{m}^3}}} = .314257\text{m} \approx 31\text{cm}\surd$$

6 Special Relativity(100pts.)

David's clock ticks once for every two of Phil's clock ticks. How fast is Phil moving by David?

Using the time dilation formula we solve for v

$$t = \frac{t_0}{\sqrt{1-\frac{v^2}{c^2}}}$$

$$\sqrt{1-\frac{v^2}{c^2}} = \frac{t_0}{t}$$

$$1-\frac{v^2}{c^2} = \left(\frac{t_0}{t}\right)^2$$

$$\frac{v^2}{c^2} = 1-\left(\frac{t_0}{t}\right)^2$$

$$v^2 = c^2\left[1-\left(\frac{t_0}{t}\right)^2\right]$$

$$v = c\sqrt{\left[1-\left(\frac{t_0}{t}\right)^2\right]} = 3\cdot 10^8\,\frac{\text{m}}{\text{s}}\sqrt{\left[1-\left(\frac{1}{2}\right)^2\right]} = 3\cdot 10^8\,\frac{\text{m}}{\text{s}}\sqrt{.75} = 2.59808\cdot 10^8\,\frac{\text{m}}{\text{s}} \approx .87c\surd$$

Phil is going 87% the speed of light.

7 Conservation of Momentum(80pts.)

You are stranded 100m away from your space shuttle. You have a 1kg physics book and decide to throw it away from the shuttle to get back. You and your space suit have a combined mass of 200kg. You only have 10min of oxygen left in your tank. How fast must you throw this book to live?

The speed you need to get back with is
$V_1 = \frac{100\text{m}}{10\text{min}} = .166667\frac{\text{m}}{\text{s}}$
At first nothing is moving so the conservation of momentum law states

$$0 = m_1 V_1 + m_2 V_2$$

Solving for the book velocity we get
$V_2 = -\frac{m_1}{m_2}V_1 = -\frac{200\text{kg}}{1\text{kg}}.166667\frac{\text{m}}{\text{s}} = 33.3334\frac{\text{m}}{\text{s}} \approx 73\text{mph}\surd$

8 Conservation of Energy(80pts.)

A 100kg skier at the top of a 2km high mountain tucks the whole thing but only gets a final speed of 50m/s. How much energy was lost to friction?

Using energy conservation the initial energy equals the final energy plus loss to nonconservative forces
$mgh = \frac{1}{2}mv^2 + E_{\text{f}}$
Solve for the energy lost to friction to get
$E_{\text{f}} = mgh - \frac{1}{2}mv^2 = 100\text{kg}\cdot 9.8\frac{\text{m}}{\text{s}^2}\cdot 2000\text{m} - \frac{1}{2}100\text{kg}\left(50\frac{\text{m}}{\text{s}}\right)^2 = 1.835\cdot 10^6\text{J} \approx 2\cdot 10^6\text{J}\surd$

9 Sound Waves(80pts.)

A man redlines his Kawasaki Ninja at 9000 rpm. You hear 12000 rpm as he drives towards you. How fast is he going?

Using the Doppler formula we solve for the speed of the source and set the observer speed to zero.

$$f = f_0 \left(\frac{v}{v \pm v_{\text{s}}} \right)$$

$$\frac{f}{f_0} = \frac{v}{v - v_{\text{s}}}$$

$$v - v_{\text{s}} = \frac{f_0}{f} v$$

$$v_{\text{s}} = v - \frac{f_0}{f} v = 343 \frac{\text{m}}{\text{s}} - \frac{900\text{rpm}}{1200\text{rpm}} 343 \frac{\text{m}}{\text{s}} = 85.75 \frac{\text{m}}{\text{s}} \approx 190\text{mph}\surd$$

10 Light Waves(100pts.)

A laser is bent as it enters water. You shine the laser at an angle of 45 degrees above a swimming pool. The laser travels 2m underwater before it hits the bottom of the pool. How deep is the pool?

First we find the angle the light is underwater using Snell's law

$$n_1 \sin \theta_1 = n_2 \sin \theta_2$$

Solve for θ_2

$$\theta_2 = \sin^{-1} \left(\frac{n_1}{n_2} \sin \theta_1 \right) = \sin^{-1} \left(\frac{1}{1.333} \frac{\sqrt{2}}{2} \right) = 32^0$$

Now using trig to find the adjacent leg(depth of pool) of this right triangle where 2m is the hypoteneuse

$$\cos 32^0 = \frac{a}{2\text{m}}$$

$$a = 1.66845\text{m} \approx 5.5\text{ft}$$