General Relativity Survey

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Abstract

General relativity (GR) is a theory of gravitation that was developed mainly by Albert Einstein between 1907 and 1915. According to general relativity, the observed gravitational attraction between masses results from their warping of space and time.

1 Introduction

By the beginning of the 20th century, Newton's law of universal gravitation had been accepted for more than two hundred years as a valid description of the gravitational force between masses. In Newton's model, gravity is the result of an attractive force between massive objects. Although even Newton was bothered by the unknown nature of that force, the basic framework was extremely successful at describing motion.

Experiments and observations show that Einstein's description of gravitation accounts for several effects that are unexplained by Newton's law, such as minute anomalies in the orbits of Mercury and other planets. General relativity also predicts novel effects of gravity, such as gravitational waves, gravitational lensing and an effect of gravity on time known as gravitational time dilation. Many of these predictions have been confirmed by experiment, while others are the subject of ongoing research. For example, although there is indirect evidence for gravitational waves, direct evidence of their existence is still being sought by several teams of scientists in experiments such as the LIGO and GEO 600 projects.

General relativity has developed into an essential tool in modern astrophysics. It provides the foundation for the current understanding of black holes, regions of space where gravitational attraction is so strong that not even light can escape. Their strong gravity is thought to be responsible for the intense radiation emitted by certain types of astronomical objects (such as active galactic nuclei or microquasars). General relativity is also part of the framework of the standard Big Bang model of cosmology.

Although general relativity is not the only relativistic theory of gravity, it is the simplest such theory that is consistent with the experimental data. Nevertheless, a number of open questions remain, the most fundamental of which is how general relativity can be reconciled with the laws of quantum physics to produce a complete and self-consistent theory of quantum gravity.

2 Mathematical Construct

Einstein's field equations is a most simple and elegant form in modern theoretical physics. Its Newtonian analogue is

$$\nabla^2\phi = 4\pi G\rho$$

where ρ is the density of mass and $c = 1$. The solution to this for a point particle mass is simply the familiar

$$\phi = -\frac{Gm}{r}$$

Since mass is not a relativistically meaningful concept, and the total energy density which includes the rest mass is only appropriate for one observer we must search for a more general meaning for the "source" of a gravitational field. The total energy density ρ is not coordinate invariant and T^{00} requires a specific reference frame. An invariant theory can avoid introducing preffered coordinate systems by using the whole of the stress energy tensor as the gravitational field source. The generalization of Newon's law would then take the form

$$\mathbf{Og} = k\mathbf{T}$$

The general form is the same, a differential operator on a solution(the metric tensor) equals the source.

3 Details of the structure

The Ricci tensor $R^{\alpha\beta}$ statisfies the conditions needed for $\mathbf{O}$. It is the only contraction of the Reimann tensor.

$$R^{\alpha\beta} + \mu g^{\alpha\beta} R = k\mathbf{T}$$

where

$$R = g^{\mu\nu} R_{\mu\nu} = g^{\mu\nu} g^{\alpha\beta} R_{\alpha\mu\beta\nu}$$

and the Riemann tensor is defined as

$$R^{\alpha}_{\beta\mu\nu} \equiv \Gamma^{\alpha}_{\beta\nu,\mu} - \Gamma^{\alpha}_{\beta\mu,\nu} + \Gamma^{\alpha}_{\sigma\mu}\Gamma^{\sigma}_{\beta\nu} - \Gamma^{\alpha}_{\sigma\sigma}\Gamma^{\sigma}_{\beta\mu}$$

where

$$\Gamma^{\mu}_{\alpha\beta}\vec{e}_{\mu} = \frac{\partial \vec{e}_{\alpha}}{\partial x^{\beta}}$$

with the interpretation that the Christoffel symbol $\Gamma^{\mu}_{\alpha\beta}$ is the μth component of $\frac{\partial \vec{e}_\alpha}{\partial x^\beta}$ making the Riemann curvature tensor above the components of the $\begin{pmatrix} 1 \\ 3 \end{pmatrix}$ tensor which, when supplied with arguements gives δV^α , return the component of the change in $\vec{V}$ on parallel transport around a loop given by $\delta a \vec{e}_\mu$ and $\delta b \vec{e}_\nu$. A flat manifold is one which has a global definition of parallelism: a vector can be moved around parallel to iteself on an arbibtrary curve and will return to its starting point unchanged. This clearly means a flat manifold is the result of $R^\alpha_{\beta\mu\nu} = 0$.

Going back to our original form we can clearly identify each term. The Einstein tensor in proportional to the stress-energy tensor.

$$G^{\alpha\beta} = 8\pi T^{\alpha\beta}$$

4 Solution Examples

4.1 The Minkowski metric

The solution to the Einstein field equations in flatspace is simply the Minkowski metric. Let's proove it. In flat space and vacuum our equation takes the form

$$R_{\alpha_\beta} - \frac{1}{2}\eta_{\alpha\beta} R = 0$$

where the Minkowski metric $\eta_{\alpha\beta}$ is the solution!
R is the Ricci Scalar defined in terms of the Riemann tensor as

$$R = R^a_{\ a} = g^{ab} R_{ab} = g^{ab} R^c_{\ acb}$$

And the Riemann Tensor is defined in terms of Christoffel symbols as

$$R^c_{\ acb} = \partial_c \Gamma^a_{bd} - \partial_b \Gamma^a_{cd} + \Gamma^e_{bc}\Gamma^a_{ed} - \Gamma^e_{bd}\Gamma^a_{ce}$$

Where the Christoffel symbols are

$$\Gamma^a_{bc} = \frac{1}{2} g^{ad} \left[\partial_c g_{db} + \partial_b g_{dc} - \partial_d g_{bc}\right]$$

Now the Christoffel symbol measures parallel transport. When two different directions are taken keeping one coordinate constant the paths may converge if the space is curved. The Christoffel symbols are a measure of this. So in flat space we get zero as the Minkowski metric has no coordinate dependence. If this is zero then the Ricci scalar is zero and the Ricci Tensor is zero satisfying the Einstein field equation for flat vacuum space.

4.2 The Schwarzchild metric

The solution to the Einstein field equations for a point mass is called the Schwarzchild metric. The result is 10 equations of motion that define the trajectory of a particle nearby this point mass. GR took a huge leap when the precession of the perhihelion of murcury was calculated when classical theory failed. In Schwarzschild coordinates, the Schwarzschild metric has the form:
$c^2 d\tau^2 = \left(1 - \frac{r_s}{r}\right) c^2 dt^2 - \left(1 - \frac{r_s}{r}\right)^{-1} dr^2 - r^2 \left(d\theta^2 + \sin^2\theta d\phi^2\right)$

- τ is the proper time (time measured by a clock moving with the particle) in seconds
- c is the speed of light in meters per second
- t is the time coordinate (measured by a stationary clock at infinity) in seconds
- r is the radial coordinate (circumference of a circle centered on the star divided by 2π) in meters
- θ is the colatitude (angle from North) in radians
- ϕ is the longitude in radians
- r_s is the Schwarzschild radius (in meters) of the massive body, which is related to its mass M by $r_s = \frac{2GM}{c^2}$, where G is the gravitational constant.

5 Gravitational Waves

In a weak field, gravity waves emerge in the theory. Then weak field Einstein equations are

$$\Box \bar{h}^{\mu\nu} = -16\pi T^{\mu\nu}$$

These are called the field equations of 'linearized theory' since they result result from keeping terms linear in $h_{\alpha\beta}$ which is a small perturbation from the Minkowski metric $\eta_{\alpha\beta}$. This is sometimes called the 'Nearly Lorentz' Coordinate system metric

$$g_{\alpha\beta} = \eta_{\alpha\beta} + h_{\alpha\beta}$$

In a vacuum,$T^{\mu\nu} = 0$, the weak field equations are

$$\left(-\frac{\partial^2}{\partial t^2} + \nabla^2\right) \bar{h}^{\alpha\beta} = 0$$

This is a 3 dimensional wave equation with solution of the form

$$\bar{h}^{\alpha\beta} = A^{\alpha\beta} e^{ik_\alpha x^\alpha}$$

This wave is a space time metric distortion packet traveling at the speed of light!