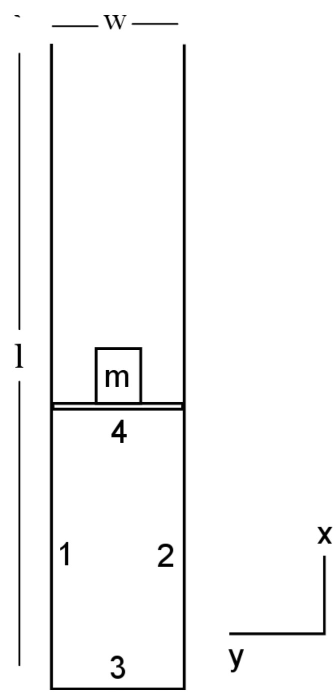
Rail Gun

January 30, 2015

Description of problem:

A projectile of mass m will be launched from a rail gun of length l , width w, resistance R. The power supply will be a capacitor bank of capacitance C at initial voltage of V_{cap}.



Application of the Biot-Savart Law:

The magnetic field generated at any given point (x, y)(the mass's location) is given by the sum of the three other rail's (1,2, and 3) contrubutions.

$$\vec{B} = \vec{B}_1 + \vec{B}_2 + \vec{B}_3$$

where

$$\vec{B}_1 = \frac{\mu_0 I}{4\pi y}\left(\frac{x}{\sqrt{x^2+y^2}} + \frac{l-x}{\sqrt{(l-x)^2+y^2}}\right)\hat{z}$$

$$\vec{B}_2 = \frac{\mu_0 I}{4\pi (w-y)}\left(\frac{l-x}{\sqrt{(l-x)^2+(w-y)^2}} + \frac{x}{\sqrt{x^2+(w-y)^2}}\right)\hat{z}$$

$$\vec{B}_3 = \frac{\mu_0 I}{4\pi x}\left(\frac{w-y}{\sqrt{x^2+(w-y)^2}} + \frac{y}{\sqrt{x^2+y^2}}\right)\hat{z}$$

For the electromotive force we include the field from the sliding current element as well

$$\vec{B}_4 = \frac{\mu_0 I}{4\pi (l-x)}\left(\frac{y}{\sqrt{y^2+(l-x)^2}} + \frac{w-y}{\sqrt{(w-y)^2+(l-x)^2}}\right)\hat{z}$$

so this defines the total as

$$\vec{B}_{\text{tot}} = \vec{B}_1 + \vec{B}_2 + \vec{B}_3 + \vec{B}_4$$

Results of Induced EMF in loop:

Faraday tells us,

$$\nabla \times \vec{E} = -\frac{d\vec{B}_{\text{tot}}}{dt}$$

.

Using Stokes theorem

$$\int \left(\nabla \times \vec{E}\right) \cdot d\vec{A} = \oint \vec{E} \cdot d\vec{l}$$

we may write

$$\int \left(-\frac{d\vec{B}_{\text{tot}}}{dt}\right) \cdot d\vec{A} = \varepsilon \rightarrow \boxed{\varepsilon(t) = -\int \frac{d\vec{B}_{\text{tot}}}{dt} dxdy}$$

This induced EMF opposes the power supply which can be written in Ohm's law

$$V(t) = I(t)\,R = V_{\text{cap}}(t) - \varepsilon(t)$$

Therefore, the current that can be used in Biot-Savart application is

$$I(t) = \frac{V_{\text{cap}}(t) - \varepsilon(t)}{R}$$

To add details, the power supply's voltage is given as

$$C = \frac{q(t)}{V_{\text{cap}}(t)} \rightarrow V_{\text{cap}}(t) = \frac{1}{C}\left[q(0) - \int I(t)\,dt\right]$$

because discharging capacitors obey

$$q(t) = q(0) - \int I(t)\, dt$$

making the current

$$I(t) = \frac{\frac{1}{C}\left[q(0) - \int I(t)\, dt\right] - \varepsilon(t)}{R}$$

The time-derivative of this is

$$\frac{dI}{dt} = \frac{d}{dt}\left(\frac{1}{RC}\left[q(0) - \int I(t)\, dt\right] - \frac{\varepsilon(t)}{R}\right) = \frac{1}{RC}\frac{d}{dt}q(0) - \frac{1}{RC}\frac{d}{dt}\int I(t)\, dt - \frac{d}{dt}\frac{\varepsilon(t)}{R}$$

$$\boxed{\frac{dI}{dt} = -\frac{I(t)}{RC} - \frac{1}{R}\frac{d\varepsilon}{dt}}$$

Force on the mass:

The Lorentz force is given by

$$\vec{F} = I(t)\int d\vec{l} \times \vec{B}_{\text{tot}} = I(t)\int_0^w \left|\vec{B}_{\text{tot}}\right| dy$$

$$\boxed{m\frac{d^2}{dt^2}x(t) = \frac{\mu_0 x(t)\, I^2(t)}{\pi}\left(\frac{w^2 + x^2(t) - x(t)\sqrt{w^2 + x^2(t)}}{\sqrt{w^2 + x^2(t)}}\right)}$$

and the equation for the current:

$$\boxed{\frac{dI}{dt} = -\frac{I(t)}{RC} + \frac{1}{R}\frac{d}{dt}\left(\int \frac{d\vec{B}_{\text{tot}}}{dt} dx dy\right)}$$

Now to solve these equations!