Assignment 2

Goldstein 2.3 Prove that the shortest distance between two points in space is a straight line.

The distance between two points is given by the integral of the infinitesimal arclength:

\[ s = \int_{1}^{2} \sqrt{dx^2 + dy^2 + dz^2} \]

where the dot represents derivative with respect to \(x\). We can perform the variation on \(s\) to find its extremum. The Lagrange equations will be

\[
0 = \frac{\partial f}{\partial y} - \frac{d}{dx} \left( \frac{\partial f}{\partial \dot{y}} \right) = 0 - \frac{d}{dx} \left( \frac{\dot{y}}{\sqrt{1 + \dot{y}^2 + \dot{z}^2}} \right)
\]

\[
0 = \frac{\partial f}{\partial z} - \frac{d}{dx} \left( \frac{\partial f}{\partial \dot{z}} \right) = 0 - \frac{d}{dx} \left( \frac{\dot{z}}{\sqrt{1 + \dot{y}^2 + \dot{z}^2}} \right)
\]

Both equations can be immediately integrated to yield:

\[
\frac{\dot{y}}{\sqrt{1 + \dot{y}^2 + \dot{z}^2}} = a \\
\frac{\dot{z}}{\sqrt{1 + \dot{y}^2 + \dot{z}^2}} = b
\]

which implies that \( \dot{z} = c\dot{y} \) where \(a, b\) and \(c\) are constants. Using this, we can write an equation just in terms of \(\dot{y}\) and solve it:

\[
\frac{\dot{y}}{\sqrt{1 + (1 + c)\dot{y}^2}} = a \\
\dot{y}^2 = \frac{a^2}{1 - a^2(1 + c)}
\]

Provided the right hand side is positive, we know \(\dot{y}\) is a constant and can be integrated to

\[ y = k_1 x + k_2 \]

which, in 3-dimensional space is an equation for a plane. The \(z\) equation can be likewise integrated with different constants:

\[ z = k_3 x + k_4 \]

which is again the equation for a plane. The intersection of the two will yield a line. Hence, the shortest distance in 3-space between two points is a line.
Goldstein 2.4 Show that the geodesics of a spherical surface are great circles, i.e., circles whose centers lie at the center of the sphere.

As with the previous problem, we need the integral of arclength to be minimized (extremized really). In this case, we are restricted to the sphere. We have

\[ s = \int_{1}^{2} ds \]
\[ = \int_{1}^{2} \sqrt{dx^2 + dy^2 + dz^2} \]
\[ = a \int_{1}^{2} \sqrt{d\theta^2 + \sin^2 \theta d\phi^2} \]
\[ = a \int_{1}^{2} \sqrt{\dot{\theta}^2 + \sin^2 \theta \, d\phi} \]

where we have used the usual transformation from Cartesian to spherical coordinates. We have used \( a \) as the radius of the sphere and allow the two generalized coordinates on the sphere to be the usual zenith and azimuthal angles, \( \theta \) and \( \phi \), respectively. Note, too, that \( \dot{\theta} \equiv d\theta/d\phi \).

Setting \( f = \sqrt{\dot{\theta}^2 + \sin^2 \theta} \), the Euler-Lagrange equation becomes

\[ 0 = \frac{d}{d\phi} \left( \frac{\partial f}{\partial \dot{\theta}} \right) - \frac{\partial f}{\partial \theta} \]
\[ = \frac{\dot{\theta}}{f} - \frac{\dot{\theta} f}{\sin \theta} \frac{1}{f^2} \sin \theta \cos \theta \]

Expanding this out, our equation becomes

\[ 0 = \dot{\theta} \sin^2 \theta - 2 \dot{\theta}^2 \sin \theta \cos \theta - \sin^3 \theta \cos \theta \]

Now come the miracles. This is a very nonlinear equation. Nonetheless, we can actually solve it. The first thing to do is to divide by the \( \sin^2 \theta \)

\[ 0 = \dot{\theta} - 2 \dot{\theta}^2 \frac{\cos \theta}{\sin \theta} - \sin \theta \cos \theta \]

We now try to combine the two derivative terms as the derivative of a single product. To do this we recall the trick of trying to find an integrating factor from first order ODEs:

\[ 0 = \frac{d}{d\phi} \left( \frac{\dot{\theta} f}{\sin^2 \theta} \right) - \frac{\dot{\theta} f}{\sin^2 \theta} \frac{1}{f} \sin \theta \cos \theta \]

where we want

\[ \frac{\dot{\theta} f}{\sin^2 \theta} = -2 \theta \frac{\cos \theta}{\sin \theta} \]

We can integrate this to get our “integrating factor” as

\[ f = \frac{C_0}{\sin^2 \theta} \]

So we have our equation as

\[ 0 = \frac{d}{d\phi} \left( \frac{\dot{\theta}}{\sin^2 \theta} \right) - \frac{\cos \theta}{\sin \theta} \]
It would now seem we are stuck except for the nonobvious observation that

$$\frac{\dot{\theta}}{\sin^2 \theta} = -\frac{d}{d\phi} \left( \frac{\cos \theta}{\sin \theta} \right)$$

so that we can write

$$0 = \frac{d^2}{d\phi^2} \left( \frac{\cos \theta}{\sin \theta} \right) + \frac{\cos \theta}{\sin \theta}$$

We have found that \( \cot \theta \) satisfies the simple harmonic oscillator equation with a frequency of 1:

$$\frac{\cos \theta}{\sin \theta} = A \cos \phi + B \sin \phi$$

This is the equation of the curve on the sphere that we want. But do we recognize it as the equation for a great circle? Probably not. I certainly didn’t. So how do we check? Well, knowing the answer ahead of time is definitely useful. Because we think the answer should be great circles, we can ask for another way to describe them. They are the intersections of planes (that pass through the origin) with spheres. We can describe origin centered spheres with the vector \( \vec{r} = a \hat{r} \) with \( a \) the fixed radius of the sphere. The equation for a plane through the origin is \( a_1 x + a_2 y + a_3 z = 0 \). It has a normal vector which describes it as \( \hat{n}_p \propto \hat{i} a_1 + \hat{j} a_2 + \hat{k} a_3 \) where we have dropped a normalization factor. The intersection of the sphere and the plane can be thought of as that set of points for which the normal to the sphere and the normal to the plane are perpendicular: \( \vec{r} \cdot \hat{n}_p = 0 \). This amounts to the equation

$$a a_1 \sin \theta \cos \phi + a a_2 \sin \theta \sin \phi + a a_3 \cos \theta = 0$$

which can be rearranged to yield

$$\cot \theta = \left( -\frac{a_1}{a_3} \right) \cos \phi + \left( -\frac{a_2}{a_3} \right) \sin \phi$$

which, except for some definitions of constants, is exactly the equation we found. Thus the shortest distance between two points (i.e., geodesics) on the sphere are great circles.
Goldstein 2.6 Find the Euler-Lagrange equation describing the brachistochrone curve for a particle moving inside a spherical Earth of uniform mass density. Obtain a first integral for this differential equation by analogy to the Jacobi integral $h$. With the help of this integral, show that the desired curve is a hypocycloid (the curve described by a point on a circle rolling on the inside of a larger circle). Obtain an expression for the time of travel along the brachistochrone between two points on Earth’s surface. How long would it take to go from New York to Los Angeles (assumed to be 4800 km apart on the surface) along a brachistochrone tunnel (assuming no friction) and how far below the surface would the deepest point of the tunnel be?

We must extremize the time of travel between two points on a sphere with the path passing through the uniformly dense sphere. In particular, we must extremize

$$T = \int \frac{ds}{v}$$

where $ds$ is the infinitesimal arclength and $v$ is the velocity along the curve. If we use spherical coordinates, it should be clear that we can rotate our system so that $\phi = 0$ along the brachistochrone path. This allows us to write $ds = \sqrt{dr^2 + r^2 d\theta^2}$.

We also need to know the velocity. We can get this from conservation of energy of a falling particle inside a sphere of uniform density. In particular, we can write

$$\frac{1}{2}mv^2 + V(r) = E$$

where the total energy, $E$, is a constant and the potential can be found from the conservative force on a particle of mass $m$ due to the gravity of some spherical mass $M(r)$:

$$F(r) = -Gm \frac{M(r)}{r^2} = -Gm \frac{4}{3} \pi r^3 \rho_0 \frac{1}{r^2} = -V'(r)$$

where $\rho_0$ is the assumed constant density of the earth. Integrating, we get

$$V(r) = \frac{1}{2} G m M_e \frac{r^2}{R_e^3} + C_0$$

in which we have used $M_e$ and $R_e$ as the mass and radius of the earth, respectively. If we assume that our particle has zero velocity at the surface of the earth, we can now solve for $v(r)$

$$v(r) = \left( \frac{G M_e}{R_e^3} \right)^{1/2} \sqrt{R_e^2 - r^2}$$

Putting it all together, we get

$$\left( \frac{G M_e}{R_e^3} \right)^{1/2} T[r(\theta)] = \int_1^2 d\theta \sqrt{\frac{\theta^2 + r^2}{R^2 - r^2}}$$

where we have reverted to using $R$ instead of $R_e$. Applying the Euler-Lagrange equations to the function

$$f(r, \dot{r}) = \sqrt{\frac{\theta^2 + r^2}{R^2 - r^2}}$$

we get (after considerable algebra)

$$0 = \left[ \frac{1}{(r^2 + \dot{r}^2)(R^2 - r^2)} \right]^{3/2} \left( \dot{r} r(R^2 - r^2) + \dot{r}^2 (r^2 - 2R^2) - r^2 R^2 \right)$$
The factor in front we can drop, recognize that \( r < R \) makes sense and despair of ever solving the resulting second order, nonlinear differential equation. However, the problem gives us a hint and notes that we can do something like the Jacobi integral, \( h \). In the mechanics case, the Jacobi, or energy, integral is a combination that is constant because it has no explicit dependance on the time parameter. Here, we actually have an analagous thing in that our function \( f \) has no explicit dependance on the coordinate \( \theta \). We can thus construct

\[
\frac{\partial f}{\partial \dot{r}} \dot{r} - f
\]

and have every confidence that it is a constant (conserved) quantity as the Jacobi integral. Doing the algebra, we get

\[
\frac{\partial f}{\partial \dot{r}} \dot{r} - f = -\frac{r^2}{(r^2 + \dot{r}^2)^{1/2}(R^2 - r^2)^{1/2}} = -C^2
\]

where this will be a constant along our curve \( r(\theta) \). Indeed, this quantity should help us get that curve. We will do that by rearranging this equation and then trying to separate variables for this first order ODE. But first, we need to identify our constant in terms of physical quantities. To this end, let us consider possible “initial” or boundary conditions. Our problem is to travel between two points on the surface of a sphere by tunneling through the actual sphere. The starting and end points will therefore be \( r(\theta_1) = R \) and \( r(\theta_2) = R \). From the symmetry of the problem, it should be clear that the lowest point along the trajectory within the sphere will come halfway between those starting points in \( \theta \). At that lowest point along the trajectory, the rate of change of \( r \) with respect to \( \theta \) will vanish, i.e. \( \dot{r}(\theta) = 0 \). We will say that this happens at \( r = r_0 \). Note that for the trajectory, \( r_0 \leq r \leq R \). Using this in the above, we have

\[
C^2 = \frac{r_0^2}{(R^2 - r_0^2)^{1/2}}
\]

Rearranging, we get

\[
\dot{r}^2 = \frac{R^2 - r^2}{r^2 - r_0^2} \frac{r^2 - r_0^2}{R^2 - r^2}
\]

Separating variables for this first order ODE we get:

\[
\frac{R}{r_0} \frac{d\theta}{\theta - \theta_0} = \int \frac{dr}{\sqrt{\frac{R^2 - r^2}{r^2 - r_0^2}}}
\]

We want to integrate this, but it looks awful. Maple or Mathematica can help, but being brave, we can also do it ourselves. Letting \( x = r/R \) and \( \epsilon = r_0/R \), we note \( \epsilon \leq x \leq 1 \). We now let

\[
v^2 = \frac{R^2 - r^2}{r^2 - r_0^2}
\]

which yields an integral of the form

\[
\frac{R}{r_0} (\theta - \theta_0) = \int dv \left[ \frac{1}{1 + v^2} = \frac{1}{1 + \epsilon^2 v^2} \right]
\]

On integrating and rearranging, we have

\[
\theta - \theta_0 = \frac{r_0}{R} \tan^{-1} \left[ \frac{\sqrt{R^2 - r^2}}{r^2 - r_0} \right] - \tan^{-1} \left[ \frac{r_0}{R} \frac{\sqrt{R^2 - r^2}}{r^2 - r_0} \right]
\]

While it is not apparent to me, this is supposedly the equation for a hypocycloid.
To get the time of travel between two points on the Earth’s surface, we return to the original quantity, $T$, to be extremized and use the path we just found between $\theta_1$ and $\theta_2$:

\[
\left(\frac{GM_e}{R^3}\right)^{1/2} T = \int_{\theta_1}^{\theta_2} \sqrt{\frac{\dot{r}^2 + r^2}{R^2 - r^2}} \, d\theta \\
= 2 \int_{r_0}^{R} \sqrt{\frac{\dot{r}^2 + r^2}{R^2 - r^2}} \, \frac{1}{r} \, dr \\
= 2 \sqrt{1 - \frac{r_0^2}{R^2}} \int_{r_0}^{R} \frac{r \, dr}{\sqrt{R^2 - r^2} \sqrt{\dot{r}^2 + r^2}}
\]

where in the second line, we have converted the integral to being with respect to $r$ instead of $\theta$. Note that we integrate from the lowest point on the trajectory to the surface and multiply that time by 2 by virtue of symmetry. The remaining integral can be evaluated by table or Mathematica or by just substituting

\[
\frac{1}{2} \left( R^2 - r_0^2 \right) u = r^2 - \frac{1}{2} \left( R^2 + r_0^2 \right)
\]

On doing this, we find

\[
T = \left( \frac{R}{GM_e} \right)^{1/2} 2 \sqrt{R^2 - r_0^2} \int_{-1}^{1} \frac{du}{\sqrt{1 - u^2}}
\]

This is the time it would take to travel between two points on the surface of the earth using this brachistochrone tunnel. Note that we have no reference to the actual coordinate values of our starting and ending points. This is tied up in the definition of $r_0$. So let's sort this out. If we say again that these points are $r(\theta_1) = R$ and $r(\theta_2) = R$, we can use our hypocycloid curve to find a relation between these and $r_0$. In particular, at $\theta_1$, we have

\[
\theta_1 - \theta_0 = \frac{r_0}{R} \tan^{-1}(0) - \tan^{-1}(\infty) = 0.
\]

Let us now orient our coordinates so that $\theta = 0$ corresponds to the lowest point on the trajectory, $r(\theta) = r_0$. We get

\[
0 - \theta_0 = \frac{r_0}{R} \tan^{-1}(\infty) - \tan^{-1}(0) = \frac{\pi}{2} \left( \frac{r_0}{R} - 1 \right)
\]

which, together with the previous, gives us

\[
\theta_1 = \frac{\pi}{2} \left( 1 - \frac{r_0}{R} \right)
\]

Putting in some numbers (i.e. $\theta_1 \approx 4800/40000/2 = 0.06$), we find that $T \approx 490$ seconds to get between New York and LA. The distance beneath the surface traveled would be about 230 km which is over halfway into the solid mantle of the earth.
Goldstein 2.13  A heavy particle is placed at the top of a vertical hoop. Calculate the reaction of the hoop on the particle by means of Lagrange’s undetermined multipliers and Lagrange’s equations. Find the height at which the particle falls off.

Using polar coordinates for the particle, \((r, \theta)\), with the origin at the center of the vertical hoop and \(\theta\) being measured from the horizontal, the Lagrangian for our particle becomes

\[
L = T - V = \frac{1}{2} m (\dot{r}^2 + r^2 \dot{\theta}^2) - m g r \sin \theta
\]

Note that we have picked the zero of potential to be the “equator” of the hoop where \(\theta = 0\). Of course, this is a constrained system so we must impose the constraint: \(f(r) = r - a = 0\) where \(a\) is the radius of the hoop. Note that technically, the constraint is \(f(r) \geq 0\) making it nonholonomic. However, we will concern ourselves only with the motion restricted to the hoop. In essence, we consider this as a holonomic system (until it isn’t). Once the particle falls off, the constraint is nonholonomic and our methods will not work. So we ask only where the constraint stops being holonomic, i.e. where the particle falls off.

Imposing the constraint is done by adding a term to Lagrange’s equations of the form

\[
0 = \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}_i} \right) - \frac{\partial L}{\partial q_i} + \lambda \frac{\partial f}{\partial q_i}
\]

where \(\lambda\) is the force of constraint, \(f\) is our constraint and if there were other forces of constraint, we would add them in an analogous manner. Using the Lagrangian above and our constraint, we get two equations:

\[
0 = m \ddot{r} - m r \dot{\theta}^2 + m g \sin \theta + \lambda
\]

\[
0 = \frac{d}{dt} (m r^2 \dot{\theta}) - m g r \cos \theta
\]

In a sense, we now impose the constraint again in that we recognize in the general equations the fact that \(r\) is not in fact changing as long as the particle is on the hoop. So we have: \(\dot{r} = \ddot{r} = 0\). This simplifies our equations to

\[
0 = -m a \dot{\theta}^2 + m g \sin \theta + \lambda
\]

\[
0 = a \ddot{\theta} - g \cos \theta
\]

The second equation can be integrated once we multiply through by \(\dot{\theta}\):

\[
C_0 = \frac{1}{2} a \dot{\theta}^2 + g \sin \theta
\]

where

\[
C_0 = \frac{1}{2} a \dot{\theta}_0^2 + g \sin \theta_0
\]

Using this, we can solve for the force of constraint:

\[
\lambda = 2 m g \left[ \frac{a}{2g} \dot{\theta}_0^2 + \sin \theta_0 - \frac{3}{2} \sin \theta \right]
\]

Note the dependance of the force of constraint on the initial conditions, \(\theta_0\) and \(\dot{\theta}_0\). When this force vanishes, we know that the particle is free of the constraint. While the problem statement is not entirely clear on this, it does seem that we can take the initial conditions as \(\theta_0 = \pi/2\) and \(\dot{\theta}_0 = 0\). In this case, we have the point where the particle flies off as

\[
\sin \theta = \frac{2}{3}
\]

or \(\theta \approx 41.8^\circ\).
Goldstein 2.14 A uniform hoop of mass $m$ and radius $r$ rolls without slipping on a fixed cylinder of radius $R$. The only external force is that of gravity. If the smaller cylinder starts rolling from rest on top of the bigger cylinder, use the method of Lagrange multipliers to find the point at which the hoop falls off the cylinder.

This problem is different from the previous one in two important aspects. The first is that we have to account for the rotational kinetic energy of the rolling hoop as well as its translational kinetic energy. Further, we will have two constraints here. To begin, we will use three coordinates. The polar coordinates, $(\rho, \theta)$, similar to those defined in the previous problem will label the center of the rolling hoop with $\theta$ measured from the horizontal. In addition, we will use the angle $\phi$ to measure the location of a fixed point around the center of the hoop. The kinetic energy of the system is then

$$T = \frac{1}{2} mr^2 \dot{\phi}^2 + \frac{1}{2} m \left[ \rho^2 + \rho^2 \dot{\theta}^2 \right]$$

where the first term on the right is the rotational part with $mr^2$ the moment of inertia of a hoop rotating about its center. The potential is

$$V = mg \rho \sin \theta$$

The Lagrangian is, of course, just $T - V$.

One constraint is that the hoop remains on the cylinder. As in the previous problem, this is holonomic until the hoop leaves the cylinder. Indeed, that is what we want to find. The constraint, again, is $f_1(\rho) = \rho - R - r = 0$. The other constraint is that the hoop rolls without slipping. This is a somewhat colloquial way of saying that the (infinitesimal) distance that the center of the hoop is translated parallel to the cylinder must be the same (infinitesimal) distance that the rim of the hoop traverses in its rotation. Mathematically, we have

$$\rho \, d\theta = r \, d\phi$$

Fortunately, this can be integrated provided the hoop remains on the sphere and becomes the holonomic constraint: $f_2(\theta, \phi) = (r + R) \theta - r \phi - C_0 = 0$ where $C_0$ is just a constant that can be ignored in the following development.

The Lagrange equations become

$$0 = \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}_i} \right) - \frac{\partial L}{\partial q_i} + \lambda_1 \frac{\partial f_1}{\partial q_i} + \lambda_2 \frac{\partial f_2}{\partial q_i}$$

where we have added two terms to accommodate the two constraints and the $\lambda$s are our forces of constraint. On using our Lagrangian we get three equations:

$$0 = m \ddot{\rho} - m \rho \dot{\theta}^2 - mg \sin \theta + \lambda_1$$

$$0 = \frac{d}{dt} (m \rho^2 \dot{\theta}) + mg \rho \cos \theta + \lambda_2 (r + R)$$

$$0 = mr^2 \ddot{\phi} - r \lambda_2$$

Importantly, the $\lambda$s can be functions of our coordinates. They are not constants. Again applying the constraints, we know that $\rho = r + R$ as long as we are on the cylinder. Hence, $\dot{\rho} = \ddot{\rho} = 0$. In addition, we can take an additional derivative of the other constraint and it must be true that $r \ddot{\phi} = \rho \ddot{\theta}$. Substituting this in the third equation above, we can solve for $\lambda_2$ and substitute this, in turn, into the second equation. We get

$$0 = \rho \ddot{\theta} + g \cos \theta + \rho \ddot{\theta}$$

which, on multiplying by $\dot{\theta}$, can be integrated to yield

$$C_1 = \dot{\theta}^2 - \frac{g}{r + R} \sin \theta$$
where the integration constant, $C_1$, can be written in terms of the initial conditions, $\theta_0$ and $\dot{\theta}_0$. With $\dot{\theta}^2$ in hand, we can substitute into the first equation and solve for the force of constraint given by $\lambda_1$:

$$\lambda_1 = 2mg \left[ \sin \theta - \frac{1}{2} \sin \theta_0 + \frac{r + R \dot{\theta}_0^2}{2g} \right]$$

This constraint, of course, represents the normal force of the cylinder on the hoop. When it goes to zero, the hoop will “fall off” the cylinder. This happens (assuming that the hoop is placed on the top of the cylinder such that $\theta_0 = \pi/2$ and $\dot{\theta}_0 = 0$) when

$$\sin \theta = \frac{1}{2}$$

or $\theta = 30^\circ$. 
A point mass is constrained to move on a massless hoop of radius $a$ fixed in a vertical plane that rotates about its vertical symmetry axis with constant angular speed $\omega$. Obtain the Lagrange equations of motion assuming the only external forces arise from gravity. What are the constants of motion? Show that if $\omega$ is greater than a critical value $\omega_0$, there can be a solution in which the particle remains stationary on the hoop at a point other than at the bottom, but that if $\omega < \omega_0$, the only stationary point for the particle is at the bottom of the hoop. What is the value of $\omega_0$?

One way to set up this problem is to view it as a one dimensional problem with the mass constrained to follow the motion of the rotating hoop. In this case, we choose a single generalized angular coordinate to describe the position of the particle on the circular hoop. Take this coordinate to be $\psi$ such that it measures the angle the mass makes from the lowest vertical position on the hoop. Now, the kinetic energy of the particle is made up of two parts. The first is its translational kinetic energy along the hoop itself. The second is rotational kinetic energy that the particle has by virtue of the rotation of the hoop. We can write these as

$$T = \frac{1}{2} m a^2 \dot{\psi}^2 + \frac{1}{2} I \omega^2$$

where $I$ is the moment of inertia of the particle with respect to the rotation axis. In this case, $I = m (a \sin \psi)^2$ where $a \sin \psi$ is the distance from the rotation axis to the particle. The potential of the particle is

$$V = m g a (1 - \cos \psi)$$

so that the zero of potential is when the particle is at the lowest point on the hoop. The Lagrangian is

$$L = \frac{1}{2} m a^2 \dot{\psi}^2 + \frac{1}{2} m a^2 \omega^2 \sin^2 \psi + m g a \cos \psi$$

where we have thrown away an irrelevant constant.

It would also be possible to get to the same point by considering the particle as moving on the surface of a sphere of radius $a$ and writing for the kinetic energy

$$T = \frac{1}{2} m a^2 (\dot{\theta}^2 + \sin^2 \theta \dot{\phi}^2)$$

where $\theta$ and $\phi$ are the usual angular coordinates of spherical polar coordinates. In the current setup, we know that the rotation in $\phi$ is a constant and is given by the angular speed, hence $\dot{\phi} = \omega$. Including the potential and equating $\psi$ with $\theta$ yields, then, the same Lagrangian.

The equation of motion from the Lagrangian thus becomes

$$0 = m a^2 \ddot{\psi} - m a^2 \omega^2 \sin \psi \cos \psi + m g a \sin \psi$$

Note that because $\partial L/\partial \dot{\psi}$ does not vanish, there is no conservation of angular momentum in the $\psi$ direction. However, the Lagrangian does not depend explicitly on $t$ so we have a Jacobi integral which is conserved:

$$h = \frac{\partial L}{\partial \dot{\psi}} \dot{\psi} - L = \frac{1}{2} m a^2 \dot{\psi}^2 - \frac{1}{2} m a^2 \omega^2 \sin^2 \psi - m g a \cos \psi$$

As we might expect, the (conserved) energy can be found by multiplying our equation of motion by $\dot{\psi}$ and integrating. On doing this, we get

$$E = T + V = \frac{1}{2} m a^2 \dot{\psi}^2 - \frac{1}{2} m a^2 \omega^2 \sin^2 \psi - m g a \cos \psi + E_0$$

where $E_0$ is an integration constant related to the zero of the potential. Note that $h$ is not the total energy, but that both quantities are conserved.
Asking for the stationary (equilibrium) points of the particle is asking where $\ddot{\psi}$ vanishes in the equation of motion, i.e.

$$0 = \omega^2 \sin \psi \left[ \cos \psi - \frac{g}{a \omega^2} \right]$$

This has solutions at $\psi = 0, \pi, \cos^{-1}(g/a\omega^2)$. From a bit of physical intuition, it should be clear that the point at $\psi = \pi$ (the top of the hoop) will be unstable while the point at $\psi = 0$ (the bottom of the hoop) will be stable. However, the equilibrium at $\cos^{-1}(g/a\omega^2)$ will only exist provided $g/a\omega^2 < 1$, or $\omega > \sqrt{g/a}$. We are inclined to call this frequency $\omega_0 = \sqrt{g/a}$, but we will need to show a few things before we completely answer the question. So for the moment, think of $\omega_0$ as a shorthand for $\sqrt{g/a}$.

With three possible equilibria, we now examine their respective stability. Considering the $\psi = 0$ solution first, let $\psi = \epsilon(t)$ where we will assume a linear expansion for $\psi$ so that $\epsilon(t)$ is always small. Substituting this into the equation of motion gives us the small amplitude motion of our particle around the equilibrium position at the bottom of the hoop:

$$0 = \ddot{\epsilon} + (\omega_0^2 - \omega^2) \epsilon$$

Note that we have used linear expansions such that $\sin \epsilon \approx \epsilon$ and $\cos \epsilon \approx 1$. The solutions to this equation depend on the sign of $\omega_0^2 - \omega^2$. Provided $\omega < \omega_0$, this quantity is positive, the solutions for the perturbations will be oscillatory and the particle will have a stable stationary point on the bottom of the hoop. However, if $\omega > \omega_0$, then the solutions will be exponentials and the perturbations will not remain small but could blow up. Hence, the particle will not have a stable equilibrium at the bottom.

Expanding $\psi$ around $\pi$ we have $\psi = \pi - \epsilon(t)$ which, when put into our equation of motion, gives

$$0 = -\ddot{\epsilon} + (\omega_0^2 + \omega^2) \epsilon$$

where, again, we have used the linear expansions $\sin(\pi - \epsilon) \approx \epsilon$ and $\cos(\pi - \epsilon) \approx -1$. This equation always has exponential solutions so we know that, in accord with our intuition, the equilibrium point for the particle and the top of the hoop is unstable.

Finally, if $\omega > \omega_0$, we have a third equilibrium point. From the problem statement, we can already conclude that this can be a stable stationary point, but let’s show this. Expanding again, but this time about $\psi_0 \equiv \cos^{-1}(\omega_0^2/\omega^2)$, we can write

$$\psi = \psi_0 + \epsilon$$

with the equation of motion becoming

$$0 = \ddot{\epsilon} - \omega^2 \sin(\psi_0 + \epsilon) \cos(\psi_0 + \epsilon) + \omega_0^2 \sin(\psi_0 + \epsilon)$$

$$= \ddot{\epsilon} - \omega^2 \left[ \sin \psi_0 + \epsilon \cos \psi_0 \right] \left[ \cos \psi_0 - \epsilon \sin \psi_0 \right] + \omega_0^2 \left[ \sin \psi_0 + \epsilon \cos \psi_0 \right]$$

$$= \ddot{\epsilon} + \epsilon \left[ \omega^2 \left( \sin^2 \psi_0 - \cos^2 \psi_0 \right) + \omega_0^2 \cos \psi_0 \right] - \omega^2 \sin \psi_0 \cos \psi_0 + \omega_0^2 \sin \psi_0$$

$$= \ddot{\epsilon} + \omega^2 \left( 1 - \frac{\omega_0^2}{\omega^2} \right) \epsilon$$

where we have used $\cos \psi_0 = \omega_0^2/\omega^2$ repeatedly and kept things to first order in $\epsilon$ only. Of course, this equation only makes sense for $\omega > \omega_0$. That being the case, the solutions are oscillatory (with frequency $\omega \sqrt{1 - \omega_0^2/\omega^2}$) and the stationary point is stable.

As a final comment, it is perhaps worth trying to get a slightly more physical feel for the condition $\omega > \omega_0 = \sqrt{g/a}$. Taking the reciprocals of both $\omega$'s give characteristic times in the problem. Of course, one is on the order of the time of revolution of the hoop while the other ($\sqrt{g/a}$) can be thought of as the time it takes to fall the distance $a$. So the condition for the stationary point to be at $\psi = \cos^{-1}(\omega/\omega_0)$ can be thought of as the condition that the period of revolution for the hoop be shorter than the freefall time for the bead across the hoop. Similarly, we can take $\omega^2$, multiply by the radius, $a$, and identify it with the centripetal acceleration associated with the motion in the $\phi$ direction. Thus the condition for the existence and stability of the stationary point becomes $a\omega^2 > g$, the angular velocity must be large enough that the centripetal acceleration will be larger than the acceleration due to gravity.
Goldstein 2.19 A particle moves without friction in a conservative field of force produced by various mass distributions. In each instance, the force generated by a volume element of the distribution is derived from a potential that is proportional to the mass of the volume element and is a function only of the scalar distance from the volume element. For the following fixed, homogeneous mass distributions, state the conserved quantities in the motion of the particle:

(a) The mass is uniformly distributed in the plane \( z = 0 \).
(b) The mass is uniformly distributed in the half-plane \( z = 0, y > 0 \).
(c) The mass is uniformly distributed in a circular cylinder of infinite length, with axis along the \( z \) axis.
(d) The mass is uniformly distributed in a circular cylinder of finite length, with axis along the \( z \) axis.
(e) The mass is uniformly distributed in a right circular of elliptical cross section and infinite length, with axis along the \( z \) axis.
(f) The mass is uniformly distributed in a dumbbell whose axis is oriented along the \( z \) axis.
(g) The mass is in the form of a uniform wire wound in the geometry of an infinite helical solenoid, with axis along the \( z \) axis.

(a) If we use Cartesian coordinates, there are translational symmetries in the \( x \) and \( y \) directions so \( p_x \) and \( p_y \) are constants. Using polar coordinates, \((\rho, \phi, z)\), we can interpret this as a rotational symmetry so \( p_\phi \) would be a constant.

(b) If we use Cartesian coordinates, there is a translational symmetry in the \( x \) direction so \( p_x \) is a constant.

(c) If we use polar coordinates, there is a translational symmetry in the \( z \) direction and a rotational symmetry in the \( \phi \) direction. Hence, \( p_z \) and \( p_\phi \) are constants.

(d) If we use polar coordinates, there is a rotational symmetry in the \( \phi \) direction so \( p_\phi \) is constant.

(e) If we use polar coordinates, there is a translational symmetry in the \( z \) direction so \( p_z \) is constant.

(f) If we use polar coordinates, there is a rotational symmetry in the \( \phi \) direction so \( p_\phi \) is constant.

(g) In this case there is no symmetry in the \( z \) or \( \phi \) directions. Therefore \( p_z \) and \( p_\phi \) are not conserved. However, if we imagine moving (translating and rotating) with the helix, it is clear that the potential remains unchanged. In particular, let the rise in the helix for each time around a loop of the helix be \( h \). Then, if we follow a path for which \( \rho \) remains constant and \( z \) changes by \( h \) for each \( 2\pi \) rotation in \( \phi \), then the combination \( p_z + hp_\phi/2\pi \) will be a conserved quantity.
Goldstein 2.20 A particle of mass \( m \) slides without friction on a wedge of angle \( \alpha \) and mass \( M \) that can move without friction on a smooth horizontal surface. Treating the constraint of the particle on the wedge by the method of Lagrange multipliers, find the equations of motion for the particle and wedge. Also obtain an expression for the forces of constraint. Calculate the work done in time \( t \) by the forces of constraint acting on the particle and on the wedge. What are the constants of motion for the system? Contrast the results you have found with the situation when the wedge is fixed.

One of the confusions that can arise in this problem is the ambiguity that can arise in picking a coordinate system. Some coordinate systems are better than others, but we will do something fairly naive. We will orient our wedge so that its rising edge has positive slope. We will say that at the initial time, the lowest corner of the wedge on the left will be at the origin of the usual \( x-y \) coordinate system and that our particle is on the upper right corner of the wedge. At a later time, after the particle and wedge have started to move, the particle’s coordinates will be \((x, y)\) with respect to the \( x-y \) coordinate system and the wedge will have moved \( X \) to the right. We will also introduce the quantity \( l \) which is the distance the particle has moved down the wedge (as measured along the sloped side of the wedge). There is, of course, a constraint which says that the particle remains on the wedge. So we have a problem with three generalized coordinates and one constraint.

The constraint can be found by looking for relations between the three generalized coordinates. In particular, at time \( t > 0 \), the bottom length of the wedge can be coordinatized as \( x - X + l \cos \alpha \) and the height of the wedge is \( y + l \sin \alpha \). The ratio of the height to the length, of course, must be \( \tan \alpha \). On simplifying this ratio, we find our constraint:

\[
f(x, y, X) = (x - X) \sin \alpha - y \cos \alpha = 0
\]

The Lagrangian can be written as

\[
L = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2) + \frac{1}{2} M \dot{X}^2 - mgy
\]

Using Lagrange multipliers, the Euler-Lagrange equations become

\[
\begin{align*}
0 &= m \ddot{x} + \lambda \sin \alpha \\
0 &= m \ddot{y} + mg - \lambda \cos \alpha \\
0 &= M \ddot{X} - \lambda \sin \alpha
\end{align*}
\]

These equations of motion can now be solved for the motion as well as the force of constraint, \( \lambda \). In particular the \( x \) and \( X \) equations can be combined to give

\[
m \ddot{x} + M \ddot{X} = 0
\]

which has an immediate solution of

\[
x (1 + \frac{m}{M}) = y \cot \alpha + \frac{1}{M} (c_0 t + c_1)
\]

with \( c_0 \) and \( c_1 \) integration constants. What this says of course, is that the center of mass moves with constant velocity, i.e. linear momentum in the horizontal direction is conserved. We should have expected this, but it is gratifying that it comes out of our analysis. With this relation, we can eliminate \( X \) from the constraint equation and view it as a relation between \( x \) and \( y \):

\[
x (1 + \frac{m}{M}) = y \cot \alpha + \frac{1}{M} (c_0 t + c_1)
\]

In addition, we can use this to write

\[
\ddot{x} (1 + \frac{m}{M}) = \ddot{y} \cot \alpha,
\]

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substitute it into the original $\ddot{x}$ equation, eliminate $\ddot{y}$ and solve for $\lambda$:

$$\lambda = -mg \frac{\cos \alpha}{1 + (m/M) \sin^2 \alpha}$$

The important point for our purposes is that $\lambda$ is a constant in time. Hence, the remaining ODEs are immediately integrable:

$$mx = -\frac{1}{2} \lambda \sin \alpha t^2 + c_2 t + c_3$$
$$my = \frac{1}{2} \left(-mg + \lambda \cos \alpha\right) t^2 + c_4 t + c_5$$

where the constants of integration will be related to the initial conditions.

Note that the forces of constraint here are embodied in $\lambda$. With regard to the constraint force of the the wedge on the particle, it is the normal force which keeps the particle on the wedge. As the wedge goes “flat” ($\alpha \to 0$), the normal force goes to $mg$ while as the wedge becomes steeper, the normal force decreases. In general, the normal force has components in both the $x$ and $y$ directions: $\vec{F}_N = |\lambda|(-\sin \alpha, \cos \alpha)$. It remains normal to the wedge for all time. It would seem that it should do no work on the particle. However, the particle’s trajectory is not perpendicular to the normal force. Because the wedge moves, the particle traces a trajectory which allows the normal force to do work on it. In particular, we can write the work done on the particle

$$W_m(t) = \int \vec{F}_N \cdot d\vec{x}_m$$
$$= \int \vec{F}_N \cdot \vec{v}_m \, dt$$
$$= |\lambda| \int \left(-\sin \alpha \dot{x} + \cos \alpha \dot{y}\right) \, dt$$
$$= |\lambda| \left[\cos \alpha y(t) - \sin \alpha x(t)\right]$$

which, if you want, can be written out in its full glory. The point, however, is that the work done by the normal force on the particle in time $t$ is not zero. However, the constraint force also does work on the wedge, namely

$$W_M(t) = \int \left(-\vec{F}_N\right) \cdot d\vec{x}_M$$
$$= -\int \vec{F}_N \cdot \vec{v}_M \, dt$$
$$= |\lambda| \sin \alpha \dot{X} \, dt$$
$$= |\lambda| \sin \alpha \left(\dot{x} - \cot \alpha \dot{y}\right) \, dt$$
$$= -W_m(t)$$

and the total work done by the forces of constraint is zero.

To find conserved quantities, we must first write our Lagrangian in terms of independent coordinates. Eliminating $X$ in the Lagrangian via the constraint relation, we have

$$L = \frac{1}{2} m(\dot{x}^2 + \dot{y}^2) + \frac{1}{2} M(\dot{x} - \cot \alpha \dot{y})^2 - mgy$$

and it is clear that $x$ is a cyclic coordinate. Hence

$$p_x = \frac{\partial L}{\partial \dot{x}} = m\dot{x} + M(\dot{x} - \cot \alpha \dot{y})$$

is a conserved quantity. This, of course, is just the (rescaled) center of mass linear momentum.
We would imagine that energy is conserved. Indeed, we can construct a Jacobi integral as our Lagrangian has no explicit dependance on time:

\[
    h = \sum_i \frac{\partial L}{\partial \dot{q}_i} \dot{q}_i - L = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2) + \frac{1}{2} M (\dot{x} - \cot \alpha \dot{y})^2 + mg y = T + V = E
\]