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Fadeev Popov Ghosts and Non-Abelian Gauge Fields

The standard model Lagrangian

$$L_{SM} = L_{YM} + L_{WD} + L_{Yu} + L_H$$

The first term, the "Yang Mills" Lagrangian, is composed of 3 familiar fields which describe the low energy gauge groups of the standard model.

$$L_{YM} = L_{QCD} + L_{I_w} + L_Y$$

The first term is invariant under the gauge group $SU(3)$ and represents strong interactions. It has an associated non-abelian gauge field but is not of importance in this discussion. The last term is invariant under the QED gauge group $U(1)$. It is a contraction of electromagnetic field strength tensors. The transformations associated with this are abelian, therefore the gauge transformation generators(just one, the identity) commute so the gauge transformation for this field does not depend on structure constants. The existence of structure constants is how we obtain nontrivial ghosts. As you can see below there are no structure constants for a $U(1)$ gauge transformation.

$$A_\mu \to A_\mu^{'} = A_\mu + \partial_\mu \Lambda$$

It's the second term in the Yang-Mills Lagrangian where we will demonstrate the power of the Fadeev-Popov method. L_{I_w} is invariant under the gauge group, $SU(2)$. It may look explicitly like this

$$L_{I_w} = \left[i\hbar c \bar{\psi}\gamma^\mu \partial_\mu \psi - mc^2 \bar{\psi}\psi \right] - \frac{1}{16\pi} \sum_{a=1}^{3} F_{\mu\nu}^a F^{\mu\nu a} - \sum_{a=1}^{3} q\bar{\psi}\gamma^\mu \tau \psi A_\mu^a$$

This describes two equal-mass, two component Dirac fields interacting with three mass less vector gauge fields(the last term). The masslessness comes from the need for a "free" gauge field term. Using the Proca Lagrangian and ignoring mass allows this. $F_{\mu\nu}^a$ is the weak isospin field strength tensor where a is three isospin vector components and an extra cross product term to keep the Lagrangian gauge invariant.

$$F^{\mu\nu} = \partial^\mu \mathbf{A}^\nu - \partial^\nu \mathbf{A}^\mu - \frac{2q}{\hbar c} \left(\mathbf{A}^\mu \times \mathbf{A}^\nu \right)$$

We will see later how this last "unfamiliar" term in this new field comes from non-abelian algebra basis generators T^a.

The last term in $\mathcal{L}_{I_w}$ is an interaction term for the gauge and Dirac fields. It is also necessary for local gauge invariance as we must use a "covariant derivative" such that the product rule term, the parameter field creates, is eliminated. For example, consider a simple real gauge parameter θ

$$\psi \to e^{i\theta} \psi \left(x_\mu \right)$$

This transformation survives the derivative ∂_μ making the Lagrangian invariant because simple phases go away when we take $\bar{\psi}\psi$. However, considering a local gauge invariance with a parameter field $\theta\left(x\right)$

$$\psi \to e^{i\theta(x)} \psi \left(x_\mu \right)$$

demands the product rule giving

$$\partial_\mu \left(e^{i\theta(x)} \psi \left(x_\mu \right) \right) = i \left(\partial_\mu \theta \left(x \right) \right) e^{i\theta(x)} \psi \left(x_\mu \right) + e^{i\theta(x)} \partial_\mu \psi \left(x_\mu \right)$$

where the first term on the RHS ruins local gauge invariance. This first term is automatically eliminated using the covariant derivative

$$\mathcal{D}_\mu = \partial_\mu + iA_\mu$$

while considering the gauge transformation $A_\mu \to A_\mu + \partial_\mu \Lambda$.

Conclusion: Yang-Mills is a non-abelian gauge theory that arises from demanding local gauge invariance.

Over-counting paths through non-abelian gauge fields

When a gauge symmetry is present, there is no procedure for selecting any one solution from a range of physically-equivalent solutions, all are related by a gauge transformation. The problem stems from the path integrals over-counting field configurations related by gauge symmetries since those correspond to the same physical state. The generating functional for quantum electrodynamics diverges,i.e.

$$Z = \int \mathcal{D}A_\mu e^{i\int \mathcal{L}dx} \to \infty$$

because we are integrating over all A_μ. The Lagrangian is invariant under the gauge transformation $A_\mu \to A_\mu + \partial_\mu \Lambda$, but this is a functional integration over all possible A_μ, including all those that are related through gauge symmetry. It is possible, however, to modify the action, such that the regular methods will be applicable by adding some additional fields which break the gauge symmetry called ghost fields. This technique is called the Faddeev-Popov procedure. *Note*: DA_μ is the invariant measure over the group of transformations and $\int DA$ is the volume of the group like $\int d\theta$ is the volume of the group of rotations in 2 dimensions.

To do this, consider re-expressing the potential four vector as

$$A_\mu \sim \bar{A}_\mu, \Lambda(x)$$

This states that each potential A_μ can be reached from some fixed $\bar{A}_\mu$ by a gauge transformation $\Lambda(x)$. Different $\bar{A}_\mu$ belong to different gauge classes. Then we can split up the generating functional

$$Z = \int \mathcal{D}A_\mu e^{iS} \sim \int \mathcal{D}\bar{A}_\mu e^{iS} \int \mathcal{D}\Lambda$$

Since the S is gauge invariant, the gauge integral goes to the right and stands for all the "over-counted" path integrals. Fadeev-Popov showed how to separate the generating functional into this exact form. Now how about the generating functional in a non-abelian gauge theory, where the action is invariant under a different gauge transformation:

$$A_\mu^U = UA_\mu U^\dagger - i(\partial_\mu U)U^\dagger$$

$$U = e^{i\Lambda^a(x)T^a}$$

and in its infinitesimal form, or limiting case for small Λ

$$A_\mu^{a'} = A_\mu^a + f^{abc}A_\mu^b\Lambda^c + \partial_\mu\Lambda^a$$

This is the gauge transformation for the $SU(2)$ gauge group in which we may admit a more general global invariance for our fields because we now have both $\psi \rightarrow U\psi$ and $\bar{\psi} \rightarrow \bar{\psi}U^\dagger$ hence $\bar{\psi}\psi$ is invariant. Also, he A_μ is no longer the electromagnetic field but the weak isospin gauge field, or Yang-Mills field. The a is for the isospin vector components, the T is the transformation generator while the Λ is the group parameter field(local) associated with group elements U from the group $SU(2)$ with anti-symmetric structure constants f^{abc}. The group generators are Hermitian matrices T^a which satisfy the commutation relations

$$T^aT^b - T^bT^a = if^{abc}T^c$$

i.e.

$$[\sigma_i, \sigma_j] = 2i\epsilon_{ijk}\sigma_k$$

where the Pauli spin matrices are in isospin space, not spin space. More specifically, we get $[T \cdot A_\mu, T \cdot \Lambda]$ where we use $(\sigma \cdot a)(\sigma \cdot b) = a \cdot b + i\sigma \cdot (a \times b)$ to get a cross product term in the gauge transformation. We will see this having a significant effect on the need for non-trivial ghosts in the next section.

A worth while example at this point is, in quantum electrodynamics we use $U(1)$ as the group of unitary transformations that commute giving us an abelian gauge field theory. The structure constants for this group are zero, hence not found in the gauge transformation itself. We will see that because of this the QED ghost particles do not interact with the QED gauge fields. With non-abelian gauges the ghost fields are a necessary computational tool in that they do interact with the gauge fields and are necessary for unitarity. However, they do not correspond to any real particles in external states: they only appear as virtual particles in Feynman diagrams.

Conclusion: The path integrals for a gauge field may be "over-counted" causing meaningless generating functionals. The gauge transformation for non-abelian gauge fields introduces a constraint, the structure constants, which produce ghost field to break the gauge symmetry for a finite theory.

The FP method

Any general gauge condition is written in the form

$$F^a \left[A_\mu\right] = 0$$

Now consider the inverse of the non-abelian gauge field propagator,

$$\triangle_F^{-1} \left[A_\mu\right] = \int \mathcal{D}U \delta \left[F^a \left[A_\mu^U\right]\right]$$

where

$$U = e^{i\Lambda^a(x)T^a} \text{ and } A_\mu^U = UA_\mu U^\dagger - i\left(\partial_\mu U\right)U^\dagger$$

The quantity $F^a \left[A_\mu^U\right]$ vanishes for some vector potentials A_μ^U and we get a delta functional

$$\delta \left[F^a \left[A_\mu^U\right]\right] = \Pi_{x^\mu,a}\delta \left[F^a \left[A\left(x^\mu\right)\right]\right]$$

A product of delta functions at each point in space time.

If we express the inverted Feynman equation as

$$1 = \triangle_F \left[A_\mu\right] \int \mathcal{D}U \delta \left[F^a \left[A_\mu^U\right]\right]$$

and insert this into the generating functional for

$$Z = \int \mathcal{D}A_\mu e^{iS}$$

we get

$$Z = \int \mathcal{D}A_\mu \triangle_F \left[A_\mu\right] \int \mathcal{D}U \delta \left[F^a \left[A_\mu^U\right]\right] e^{iS}$$

Now perform a gauge transformation $A_\mu^U \to A_\mu$ to get

$$Z = \int \mathcal{D}U \int \mathcal{D}A_\mu \triangle_F \left[A_\mu\right] \delta \left[F^a \left[A_\mu\right]\right] e^{iS}$$

We have therefore isolated the term that gets "over-counted". We have broken the gauge symmetry. $\int \mathcal{D}U$ contributes only an overall multiplicative factor so we ignore it since only the normalized form of Z has physical consequences. Therefore our generating functional is

$$Z = \int \mathcal{D}A_\mu \triangle_F \left[A_\mu\right] \delta \left[F^a \left[A_\mu\right]\right] e^{iS}$$

Now we search for an explicit expression for the gauge field propagator $\triangle_{\mathrm{F}}[A_\mu]$, for then we may use the delta functional explicitly. To do this we need some specific tools such as

$$\triangle_{\mathrm{F}}[A_\mu] = \det\left[\frac{\delta F}{\delta \Lambda}\right]_{F=0} = \det M$$

which comes from the Jacobian for a change of variables.

Here is where the structure constants play their role. The determinant of the functional derivative of the Yang-Mills gauge with respect to the group parameter field is not just a constant, it contains A_μ! *Note:* With the Lorentz gauge we do get zero, hence no coupling of the gauge field with a ghost field.

Change M to iM and use

$$\int \mathcal{D}\bar{\alpha}\mathcal{D}\alpha e^{-\bar{\alpha}A\alpha} = \det A$$

in the form

$$\int \mathcal{D}\bar{\eta}\mathcal{D}\eta e^{-i\int \bar{\eta}^a M_{ab}\eta^b dx} = \det iM$$

where the η is a Grassman scalar fermionic field, a ghost field giving us

$$Z = \int \mathcal{D}A_\mu \int \mathcal{D}\bar{\eta}\mathcal{D}\eta e^{-i\int \bar{\eta}^a M_{ab}\eta^b dx}\delta\left[F^a[A_\mu] - C^a\right]e^{iS}$$

For convenience we modify the gauge fixing term with C^a , independent of A_μ. This makes Z independent of $C(x)$. We may then just multiply Z by an overall phase factor, again, since only the normalized form has physical significance.

$$Z = \int \mathcal{D}A_\mu\mathcal{D}\bar{\eta}\mathcal{D}\eta e^{-i\int \bar{\eta}^a M_{ab}\eta^b dx}e^{-\frac{i}{2\alpha}\int C_a^2(x)dx}\delta\left[F^a[A_\mu] - C^a\right]e^{iS}$$

$$= \int \mathcal{D}A_\mu\mathcal{D}\bar{\eta}\mathcal{D}\eta e^{-i\int \bar{\eta}^a M_{ab}\eta^b dx}e^{-\frac{i}{2\alpha}\int F^2(x)dx}e^{iS}$$

Allowing us to say

$$Z = N\int \mathcal{D}A_\mu d\bar{\eta}d\eta e^{-i\int\left(\mathcal{L}-\frac{1}{2\alpha}F^2-\bar{\eta}^a M_{ab}\eta^b\right)dx}$$

or

$$Z = N\int \mathcal{D}A_\mu d\bar{\eta}d\eta e^{-i\int \mathcal{L}_{\mathrm{eff}}dx}$$

where

$$\mathcal{L}_{\mathrm{eff}} = \mathcal{L} + \mathcal{L}_{\mathrm{GF}} + \mathcal{L}_{\mathrm{FPG}}$$

a more explicit form of the Fadeev-Popov ghost term is

$$\mathcal{L}_{\mathrm{FPG}} = \partial_\mu\bar{\eta}^a\partial^\mu\eta^a + gf^{abc}\left(\partial^\mu\bar{\eta}^a\right)A_\mu^b\eta^a$$

Here we can see that in an abelian gauge field we would get ghosts but not any ghosts that interact with our fields. The second term is crucial for breaking symmetry in a non-abelian field theory for it includes the interactions with the Yang-Mills field. These Faddeev-Popov ghosts are peculiar. They violate the spin-statistics relation, which is another reason why they are often regarded as "non-physical" particles. For example, in Yang-Mills theories (such as quantum chromodynamics) the ghosts are complex scalar fields (spin 0), but they anti-commute (like fermions). *In general, anti-commuting ghosts are associated with bosonic symmetries, while commuting ghosts are associated with fermionic symmetries.*

Appendix

Topics

1. Gauge Fields
2. Jacobian for the FP determinant
3. Quicksters
4. Invariant Gauge field propagator
5. Scalar Fields
6. Free Dirac Spinor Fields
7. Other Fields
8. Stem From
9. In field Theory
10. Isospin
11. Feynman Diagram appearance example

Gauge Fields

Gauge fields exploit mathematical arbitrariness in the solution to a field equation. For a classical field for example, the scalar and vector potential fields may posses a gauge term which may not effect the validity of the fields as solutions to the Maxwell equations.

$$\Phi \to \Phi' = \Phi + \frac{\partial \theta}{\partial t}$$

$$\vec{A} \to \vec{A}' = \vec{A} + \nabla \theta$$

or

$$A_\mu \to A_\mu + \partial_\mu \Lambda$$

Jacobian for the FP determinant

Consider a gauge condition f with gauge θ.

$$\triangle (r)^{-1} = \int \delta (f (\theta)) \, d\theta$$

$$= \int \delta (f (\theta)) \det \left| \frac{d\theta}{df} \right| df$$

$$= \det \left| \frac{d\theta}{df} \right|_{f=0}$$

hence

$$\triangle (r) = \det \left| \frac{df}{d\theta} \right|_{f=0}$$

Quicksters

2 types of groups, Gauge and Symmetry

$$U (1) \exists e^{i\phi 1}$$

1 is the "T^a" so there are no structure constants for $U(1)$

$\Lambda (x_\mu)$ is a parameter field for the group of gauge transformations U

T^agenerates the group or the algebra basis

<u>EXAMPLE</u>

For

$SO(3)$ or $SU(2)$

$\Lambda^1, \Lambda^2, \Lambda^3$ are the 3 angles for the three generators L_x, L_y, L_z

$$U A_\mu U^{-1} \simeq (1 + i\Lambda) A_\mu (1 - i\Lambda)$$

For $U(1)$ there is only one generator, the identity, which commutes with everything, therefore, no structure constants.

Invariant Gauge field propagator

Let us now look at how $\triangle_F^{-1} [A_\mu]$ is gauge invariant. Consider it under another gauge transformation

$$\triangle_F^{-1} \left[A_\mu^{U'} \right] = \int \mathcal{D}U \delta \left[F^a \left[A_\mu^{U'U} \right] \right]$$

Now we can stay consistent by saying $U'' = U'U$ and use the results for compact groups that the volume element in group space defines an invariant measure.

$$\triangle_F^{-1} \left[A_\mu^{U'} \right] = \int \mathcal{D}U'' \delta \left[F^a \left[A_\mu^{U''} \right] \right] = \triangle_F^{-1} \left[A_\mu^{U} \right]$$

Scalar Fields

The Klein-Gordon Lagrangian for a free scalar spin-0 field is

$$\mathcal{L} = \frac{1}{2}\left(\partial_\mu \phi\right)\left(\partial^\mu \phi\right) - \frac{1}{2}\left(\frac{mc}{\hbar}\right)^2 \phi^2$$

Gives rise to the Klein-Gordon equation when applied to the Euler-Lagrange equations

$$\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_i)}\right) = \frac{\partial \mathcal{L}}{\partial \phi_i}$$

KG is therefore

$$\partial_\mu \partial^\mu \phi + \left(\frac{mc}{\hbar}\right)^2 \phi = 0$$

vacuum to vacuum transition amplitude in the presence of a source J

$$Z\left[J\right] = \int \mathcal{D}\phi e^{i\int d^4x\left(\mathcal{L}(\phi) + J(x)\phi(x) + \frac{i}{2}\varepsilon\phi^2\right)}$$

Free Dirac Spinor Fields

The Dirac Lagrangian for a free spinor spin-1/2 field is

$$\mathcal{L} = i\left(\hbar c\right)\bar{\psi}\gamma^\mu \partial_\mu \psi - mc^2 \bar{\psi}\psi$$

Gives rise to the Dirac equation when applied to the Euler-Lagrange equations

$$\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_i)}\right) = \frac{\partial \mathcal{L}}{\partial \phi_i}$$

Dirac equation is

$$i\gamma^\mu \partial_\mu \psi - \frac{mc}{\hbar}\psi = 0$$

vacuum to vacuum transition amplitude in the presence of a source η

$$Z_0\left[\eta, \bar{\eta}\right] = \frac{1}{N}\int \mathcal{D}\bar{\psi}\mathcal{D}\psi e^{i\int\left(\bar{\psi}(x)(i\gamma\cdot\partial - m)\psi(x) + \bar{\eta}(x)\psi(x) + \bar{\psi}(x)\eta(x)\right)dx}$$

$$N = \int \mathcal{D}\bar{\psi}\mathcal{D}\psi e^{i\int \bar{\psi}(x)(i\gamma\cdot\partial - m)\psi(x)dx}$$

Other Fields

The Proca Lagrangian for a free Vector Spin-1 field is

$$\mathcal{L} = -\frac{1}{16\pi}F^{\mu\nu}F_{\mu\nu} + \frac{1}{8\pi}\left(\frac{mc}{\hbar}\right)^2 A^\nu A^\nu$$

when applied the the Euler Lagrange equations gives us the Proca Field equation

$$\partial_\mu F^{\mu\nu} + \left(\frac{mc}{\hbar}\right)^2 A^\nu = 0$$

The Maxwell Lagrangian for a Mass-less Vector Spin-1 field with source J^μ is

$$\mathcal{L} = -\frac{1}{16\pi}F^{\mu\nu}F_{\mu\nu} + \frac{1}{c}J^\mu A_\mu$$

when applied the the Euler Lagrange equations gives us the Tensor form of Maxwell equations

$$\partial_\mu F^{\mu\nu} = \frac{4\pi}{c}J^\nu$$

Stem From

When $\mathcal{H}$ is quadratic in the velocities we have the generic transition amplitudes seen above as

$$\langle q_f t_f | q_i t_i \rangle = N \int \mathcal{D}q e^{\frac{i}{\hbar} \int_{t_i}^{t_f} L(q,\dot{q})dt}$$

In field Theory

It is true in field theory that the only safe path integral involves

$$\int \int \mathcal{D}\phi \mathcal{D}\pi e^{i \int (\dot{\phi}\pi - \mathcal{H})}$$

After performing the integration of π the simple Lagrangian path integral is not recovered unless there are no constraints and the $\mathcal{H}$ is quadratic in the π with constant coefficients. Non-Abelian gauge theories do not satisfy this condition. The Fadeev-Popov method gets one out of trouble.

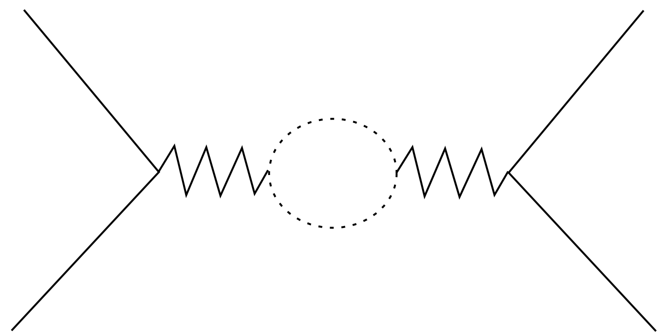
Isospin

This is a number for strong interactions, for example, how do the neutron and proton spins differ when inside the nucleus?..different isospin because of different quark composition.

i.e. The pion has 3 versions where the isospin is the only degree of freedom that is different, π^0, π^-, π^+

Feynman diagram appearance example

One of the four diagrams in QCD contributing to $q\bar{q} \to q\bar{q}$ to order g^4 is



Where the dotted lines are ghost propagators and the wavy lines are gluons, making the real external lines a quark and an anti-quark.